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Exact Orbits of Light Rays Reflected Inside Ellipses, Traced by Means of Rational Formulae with the Help of the CAS Derive™ 6

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Abstract

Presenting high school or college students with the mathematical description of billiards behavior in an elliptical domain will undoubtedly inspire them to tackle even more challenging problems over time. The development of the related algorithms will also give the students an excellent opportunity to consciously use suitable Computer Algebra Systems (CAS). In particular, the implementation of calculations in exact rational arithmetic and the management of integer numbers of arbitrarily large dimensions, with an appropriate CAS, will help the interested parties to discriminate some statements concerning the delicate issue of the possible periodicity of the trajectories generated by the subsequent reflections of the billiards on the boundaring ellipses.

1. Introduction

The purpose of this article is to exactly describe orbits in an elliptical billiard, necessarily by using "rational formulae". The use of trigonometry is therefore excluded, in order not to run the risk of introducing irrational, non-algebraic, elements. The only tools allowed are then those of algebra and analytic geometry, accessible since from the first years of upper secondary school.

The reflection of light rays inside ellipses is thoroughly investigated, with three different technological tools, GeoGebra, Maxima and Maple, in [1]. Much weight is given to the question of the periodicity of the trajectories generated by the successive reflections of light beams on the boundary of ellipses. However, only floating-point calculations are considered. Two vertices of an orbit in an ellipse are considered as the same one if "there is enough computational evidence" that some conditions of proximity are satisfied. One should not yield to the limitations of floating-point calculations but should only rely on exact results before endorsing his hypotheses.

2. Preliminaries - Description of the problem

Speaking of billiards, it would be appropriate to use the language of dynamics and consider material marbles, perhaps point-like, which move freely, with constant speed, on the green cloth of a horizontal plane table, in a region bounded by a closed curve, and which bounce infinitely many times on the perfectly elastic edges of the region, with reflection angles equal to their incidence angles, without any loss of kinetic energy.

What matters are only the laws of reflection, therefore, in order not to have to make too many concessions with respect to reality, it is preferable to adopt the language of optics, surely simpler, with its fundamental laws, of a purely geometric nature, of the rectilinear propagation of light rays in a homogeneous medium, of their independence and of their reflection on a specular surface (or straight line).

As a further simplification, it is useful to consider light rays as simple straight lines. It is of course a mathematical abstraction, chosen to facilitate reasoning, which allows to clearly represent experimental phenomena and devices: geometric lines, unlike light rays, have no thickness. Therefore, a convenient representation of the elliptical "billiard" is an ideal light ray which is reflected multiple times on the wall of a flat two-dimensional elliptical cavity. The sequence of incident and reflected rays will be called an "orbit".

In geometric language, the law of reflection is stated as follows: the angle of reflection is equal to the angle of incidence: $r = i$. The reflected and incident rays belong to a plane passing through the perpendicular (normal) straight line to the reflecting surface at the point of incidence, where they form equal angles with the normal line. If the incident ray coincides with the normal to the mirror, with an angle of incidence equal to zero, the reflected ray also forms a null reflection angle: the reflected ray coincides with the incident ray (normal incidence). This law also applies if the surface is curved. In this case the normal at the point of incidence is the perpendicular to the straight-line tangent to the surface at that point.

I believe there is no school book of mathematics, or physics, which, at the appropriate stage, does not deal with the focal properties of conics, for light rays or their extensions that pass through the foci and whose trajectory is only subject to the laws of reflection. It is also well known that in an elliptical billiard table an orbit that passes through one focus after reflection also passes through the other focus and tends to get closer and closer to the major axis of the ellipse. Here we consider instead the neglected rays, those that do not pass through the foci.

3. The incident ray

A light ray, without thickness, with the properties of a Euclidean geometric straight line, is projected inside a flat two-dimensional elliptical cavity through a hole, represented by a known point $P_j = (x_j, y_j)$ of an ellipse E, which delimits the cavity and whose equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ , \qquad \text{with } a > b > 0 \ .$$

The light ray hits the inner wall of the elliptical cavity at a point $P_f = (x_f, y_f)$, necessarily different from $P_j = (x_j, y_j)$. The direction of the light ray can be represented by a non-identically null vector $\boldsymbol{v}_j = (v_{jx}, v_{jy})$.

The equation of the straight line s representing the light ray, passing through the point $P_j = (x_j, y_j)$ of the ellipse E with direction $\boldsymbol{v}_j = (v_{jx}, v_{jy})$, can be written, in implicit form, in this way:

$$v_{j_x} \cdot (y - y_j) - v_{j_y} \cdot (x - x_j) = 0 \ .$$

The algebraic calculations necessary to obtain the second intersection P_f of the straight line s with the ellipse E are lengthy and convoluted, so much so that one can feel discouraged in attempting to bring them to an end. Some steps are really challenging. The complexity of the calculations, however, represents an excellent opportunity to enable students to consciously use an appropriate CAS, in a context that is not artificial and useless but indispensable and decisive, and can also stimulate individual or team competitions among students to get to the result.

The coordinates of the point P_f, where the straight line s intersects the ellipse E are then:

$$x_f = \frac{a^2 \cdot v_{j_y}{}^2 - b^2 \cdot v_{j_x}{}^2}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2} \cdot x_j - \frac{2 \cdot a^2 \cdot v_{j_x} \cdot v_{j_y}}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2} \cdot y_j \ , \quad (1.a)$$

$$y_f = -\frac{2 \cdot b^2 \cdot v_{j_x} \cdot v_{j_y}}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2} \cdot x_j - \frac{a^2 \cdot v_{j_y}{}^2 - b^2 \cdot v_{j_x}{}^2}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2} \cdot y_j \ . \quad (1.b)$$

In a comparable form, the previous formulae were also used in [2].
If one sets:

$$B_{11} = \frac{a^2 \cdot v_{j_y}{}^2 - b^2 \cdot v_{j_x}{}^2}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2} \qquad B_{12} = -\frac{2 \cdot a^2 \cdot v_{j_x} \cdot v_{j_y}}{a^2 \cdot v_{j_y}{}^2 + b^2 \cdot v_{j_x}{}^2}$$

$$B_{21} = \frac{b^2}{a^2} B_{12} \qquad B_{22} = -B_{11}$$

named $\mathbf{B}_{\boldsymbol{v}_j}$ the following matrix, function only of a, b and of $\boldsymbol{v}_j = (v_{j_x}, v_{j_y})$:

$$\mathbf{B}_{\boldsymbol{v}_j} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} ,$$

one can easily verify the relation:

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} ,$$

which can be concisely written like this:

$$P_f = \mathbf{B}_{\boldsymbol{v}_j} P_j \ ,$$

which abridges the formulae (1.a), (1.b).

The elements of $\mathbf{B}_{\boldsymbol{v}_j}$ depend only on the orientation chosen for the vector $\boldsymbol{v}_j = (v_{j_x}, v_{j_y})$ but not on its direction or its magnitude: the elements are in fact invariant under the simultaneous exchange of sign of both components of $\boldsymbol{v}_j$ or by their multiplication by an arbitrary factor k, because both the numerators and the denominators of the fractions defining the elements of $\mathbf{B}_{\boldsymbol{v}_j}$ are all homogeneous polynomials of the 2nd degree in the components of $\boldsymbol{v}_j$.

One can easily verify that $\mathbf{B}_{\boldsymbol{v}_j}$ is an involutory matrix:

$$\mathbf{B}_{\boldsymbol{v}_j} \mathbf{B}_{\boldsymbol{v}_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2 \ ,$$

having denoted by $\mathbf{I}_2$ the 2×2 identity matrix.

The correspondence established between the points P_j and P_f of the ellipse is an involutory correspondence: if, with a chosen direction $\boldsymbol{v}_j$, a point P_f corresponds to a point P_j, when the same direction $\boldsymbol{v}_j$ is maintained, the point P_j corresponds to the point P_f. In fact:

$$\mathbf{B}_{\boldsymbol{v}_j} P_f = \mathbf{B}_{\boldsymbol{v}_j} (\mathbf{B}_{\boldsymbol{v}_j} P_j) = \mathbf{B}_{\boldsymbol{v}_j} \mathbf{B}_{\boldsymbol{v}_j} P_j = \mathbf{I}_2 P_j = P_j \ ,$$

then: $P_f = \mathbf{B}_{\boldsymbol{v}_j} P_j$ and $P_j = \mathbf{B}_{\boldsymbol{v}_j} P_f$.

4. The reflected ray

To the direction $\boldsymbol{v}_j = (v_{jx}, v_{jy})$ of the incident ray, that is to the direction of the straight line s, it is now necessary to match the direction $\boldsymbol{v}_f = (v_{fx}, v_{fy})$ of the straight line which stands for the reflected ray going through the point $P_f = (x_f, y_f)$, already found out by means of P_j and $\boldsymbol{v}_j$.

By means of the symmetry properties of $\boldsymbol{v}_f$ and $\boldsymbol{v}_j$ with respect to the straight line p perpendicular, in the point $P_f = (x_f, y_f)$, to the tangent line t to the ellipse in the same point $P_f = (x_f, y_f)$, the direction $\boldsymbol{v}_f = (v_{fx}, v_{fy})$ of the reflected ray is established by the following formulae:

$$v_{f_x} = \frac{a^4 \cdot y_f^{\,2} - b^4 \cdot x_f^{\,2}}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}} \cdot v_{j_x} - \frac{2 \cdot a^2 \cdot b^2 \cdot x_f \cdot y_f}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}} \cdot v_{j_y} , \quad (2.a)$$

$$v_{f_y} = -\frac{2 \cdot a^2 \cdot b^2 \cdot x_f \cdot y_f}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}} \cdot v_{j_x} - \frac{a^4 \cdot y_f^{\,2} - b^4 \cdot x_f^{\,2}}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}} \cdot v_{j_y} . \quad (2.b)$$

In a comparable form, the previous formulae were also used in [2].

If one sets:

$$R_{11} = \frac{a^4 \cdot y_f^{\,2} - b^4 \cdot x_f^{\,2}}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}} \qquad R_{12} = -\frac{2 \cdot a^2 \cdot b^2 \cdot x_f \cdot y_f}{a^4 \cdot y_f^{\,2} + b^4 \cdot x_f^{\,2}}$$

$$R_{21} = R_{12} \qquad R_{22} = -R_{11}$$

named $\mathbf{R}_{P_f}$ the following matrix, function only of the point $P_f = (x_f, y_f)$:

$$\mathbf{R}_{P_f} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} ,$$

one can easily verify the relation:

$$\begin{pmatrix} v_{f_x} \\ v_{f_y} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} v_{j_x} \\ v_{j_y} \end{pmatrix} ,$$

which can be concisely written like this:

$$\boldsymbol{v}_f = \mathbf{R}_{P_f}\, \boldsymbol{v}_j ,$$

which abridges the formulae (2.a), (2.b).

The elements of $\mathbf{R}_{P_f}$ depend only on the point P_f but not at all on the vector $\boldsymbol{v}_j = (v_{jx}, v_{jy})$: in fact the matrix $\mathbf{R}_{P_f}$ produces the desired effect whatever the vector $\boldsymbol{v}_j$ assigned.

One can easily verify that, like $\mathbf{B}_{\boldsymbol{v}_j}$, also $\mathbf{R}_{P_f}$ is an involutory matrix:

$$\mathbf{R}_{P_f} \cdot \mathbf{R}_{P_f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2 .$$

The correspondence established between the directions of the vectors $\boldsymbol{v}_j$ and $\boldsymbol{v}_f$ is an involutory correspondence: if, for a chosen bounce point $P_f = (x_f, y_f)$, $\boldsymbol{v}_f$ is a vector with the direction of the ray into which an incident ray with the direction $\boldsymbol{v}_j$ is reflected, changing the roles, $\boldsymbol{v}_j$ is, reciprocally, a vector with the direction of the ray into which an incident ray with the direction $\boldsymbol{v}_f$ is reflected. In fact:

$$\mathbf{R}_{P_f}\, \boldsymbol{v}_f = \mathbf{R}_{P_f} (\mathbf{R}_{P_f}\, \boldsymbol{v}_j) = \mathbf{R}_{P_f}\, \mathbf{R}_{P_f}\, \boldsymbol{v}_j = \mathbf{I}_2\, \boldsymbol{v}_j = \boldsymbol{v}_j ,$$

then: $\boldsymbol{v}_f = \mathbf{R}_{P_f}\, \boldsymbol{v}_j$ and $\boldsymbol{v}_j = \mathbf{R}_{P_f}\, \boldsymbol{v}_f$.

5. The orbit

The procedure to follow to trace an orbit in an elliptical billiard is by now well defined.

Chosen $P_j = (x_j, y_j)$ and $\boldsymbol{v}_j = (v_{jx}, v_{jy})$, one easily determines $P_f = (x_f, y_f)$:

$$P_f = \mathbf{B}_{\boldsymbol{v}_j} P_j \ .$$

Then the direction of the reflected ray $\boldsymbol{v}_f = (v_{fx}, v_{fy})$ is also determined:

$$\boldsymbol{v}_f = \mathbf{R}_{P_f} \boldsymbol{v}_j \ .$$

The procedure is then iterated by taking P_f as a new P_j and $\boldsymbol{v}_f$ as a new $\boldsymbol{v}_j$. Practically, however, it is more convenient to chose two points on the ellipse E , to denote them P_j and P_f , and then to define $\boldsymbol{v}_j = (v_{jx}, v_{jy}) = \overline{P_j P_f}$, that is to define $v_{jx} = x_f - x_j$ and $v_{jy} = y_f - y_j$.

It is fitting to express the algorithm in a formal way. Given the initial values $P_0 = (x_0, y_0)$ and $\boldsymbol{v}_0 = (v_{0x}, v_{0y})$, for $n = 0, 1, 2, \dots$, the orbit in the elliptical billiard is described by the coupled sequence:

$$P_{n+1} = \mathbf{B}_{\boldsymbol{v}_n} P_n \ , \qquad (3.a)$$

$$\boldsymbol{v}_{n+1} = \mathbf{R}_{P_{n+1}} \boldsymbol{v}_n \ . \qquad (3.b)$$

6. The rational character of the formulae

For rational lengths of the semi-axes a and b of the ellipse E , in particular, and also for values of a and b which give rational a^2 and b^2, the above formulae are wholly rational. Therefore they carry on the numerical character of P_0 and $\boldsymbol{v}_0$. In particular, *if* the point $P_0\,(x_0, y_0)$ of the ellipse E is rational and also the vector $\boldsymbol{v}_0\,(v_{0x}, v_{0y})$, which represents the direction of the light ray that entered the ellipse from the point P_0, is rational, *then* $P_1 = \mathbf{B}_{\boldsymbol{v}_0} P_0$ and $\boldsymbol{v}_1 = \mathbf{R}_{P_1} \boldsymbol{v}_0$ are rational and consequently all the points P_n and all the vectors $\boldsymbol{v}_n$ are rational: all the points touched by the orbit on the ellipse E and the directions of all incident and reflected light rays are rational. The same qualitative property is maintained if some elements in P_0 and $\boldsymbol{v}_0$ are algebraic or transcendental irrational numbers: the rational formulae carry on the algebraic or transcendental irrational character of P_0 and $\boldsymbol{v}_0$.

7. The implementation of the algorithm

The technological tool chosen to carry out this study is the CAS Derive™ 6. It was developed by Soft Warehouse in Honolulu, Hawaii. The first release was in 1988 for DOS. Subsequently owned by Texas Instruments, but Derive was discontinued on June 29, 2007 [3]. The last and definitive version is Derive 6.1 for Windows. Despite being discontinued, Derive 6.1 can still be found and downloaded for free from a Web search [4]. The author also easily installed it on a 32-bit PC running Windows 10. Furthermore, the interested reader can find and download a treasure trove of very useful material from the International Derive User Group site [5].

By default, Derive uses its exact precision mode and rational notation, that is, it displays results using integers, fractions, and symbols, for irrational constants, such as e or π , and radicals, such as $\sqrt{m}$ for a square root. In relation to the purposes of this article, the most interesting feature of Derive is its ability to perform calculations in exact rational arithmetic, that is its capability to manage integers of arbitrarily large size by allocating them memory space dynamically, using as many bytes as needed for each number. Fractions can then be handled as such, without replacing them with rounded decimal values. All required rational values can be calculated exactly.

The exact results will only be truncated, to a suitable number of digits, for the purpose of their graphical representation. In other words, all results are rounded after the calculations and not before or during them, as inevitably happens when using only floating-point numbers.

A limitation of Derive, like comparable CAS, is the amount of the globally available memory, but this is usually not a problem, even with ordinary computers.

The algorithm implemented in Derive to plot the orbits in an elliptical billiard is presented in Figure 1. It faithfully conforms to the formulae (1.a), (1.b), (2.a) and (2.b). The recurring procedure *P*(*n*) must obviously be preceded by the assignments of the numeric values to the semi-axes of the ellipse and of the initial values to the variables, denoted by x_0, y_0, v_{0x} and v_{0y} . After having assigned the desired initial values to the four variables, then, for each choice of the index *k*, the "simplification" of the recurring procedure *P*(*n*) gives the coordinates of the *k*-th point of the orbit.

In order to calculate the coordinates of a vertex, the procedure retraces the coordinates of all the previous vertices, from the beginning: this time-consuming choice protects against mistakes and inattention in the assignment of initial or intermediate values. Moreover, if the results grow too quickly, it can be difficult to do reassignments by hand, while the algorithm does it very well on its own, one just needs to be patient. It is the price to pay to achieve exact results. From my point of view, time of computation is not a factor which needs to be taken into consideration. The task can be distributed among all the students of a class, assigning to each of them a different iteration "*k*". Then, the students can contribute with their own results, in a simplified portable form which will be dealt with later (the true points *P*(*n*) will be substituted with more manageable rational points *Q*(*n*) extremely near to them and almost indistinguishable from them). The teacher collates all the points *Q*(*n*) in a single file in order to obtain the complete graphical representation of the orbit in the ellipse, as shown in Figure 3.

```
P(n) :=
  Prog
    n := 0
    xj := x0
    yj := y0
    P(n) := [x0, y0]
    vjx := v0x
    vjy := v0y
    Loop
      If n = k
         RETURN P(n)
      xf := ((a^2·vjy^2 - b^2·vjx^2)·xj + (- 2·a^2·vjx·vjy)·yj)/(a^2·vjy^2 + b^2·vjx^2)
      yf := ((- 2·b^2·vjx·vjy)·xj - (a^2·vjy^2 - b^2·vjx^2)·yj)/(a^2·vjy^2 + b^2·vjx^2)
      P(n) := [xf, yf]
      vfx := (a^4·yf^2 - b^4·xf^2)·(xf - xj) + 2·a^2·b^2·xf·yf·(yj - yf)
      vfy := (a^4·yf^2 - b^4·xf^2)·(yj - yf) + 2·a^2·b^2·xf·yf·(xj - xf)
      xj := xf
      yj := yf
      vjx := vfx
      vjy := vfy
      n := n + 1
```

Figure 1. *Core of the procedure that allows to trace orbits in an elliptical billiard, by means of rational formulae, implemented in* Derive. See [S1] ABeJMT_1.

One may observe that in the formulae (1.a), (1.b), (2.a) and (2.b) there are some repetitive calculations. The recurring procedure *P*(*n*) can be rendered more efficient by assigning the result of the repetitive calculations to temporary variables and then use their values when needed. So, in each cycle of the loop, a number of unnecessary operations are spared.

The modified algorithm, implemented in Derive, is presented in Figure 2.

```
P(n) :=
  Prog
    n := 0
    xj := x0
    yj := y0
    P(n) := [x0, y0]
    vjx := v0x
    vjy := v0y
    a2 := a^2
    a4 := a2^2
    b2 := b^2
    b4 := b2^2
    da2b2 := 2·a2·b2
    Loop
      If n = k
         RETURN P(n)
      vjx2 := vjx^2
      vjy2 := vjy^2
      coeff1 := a2·vjy2 - b2·vjx2
      coeff2 := - 2·vjx·vjy
      den := a2·vjy2 + b2·vjx2
      xf := (coeff1·xj + coeff2·a2·yj)/den
      yf := (- coeff1·yj + coeff2·b2·xj)/den
      P(n) := [xf, yf]
      coeff3 := a4·yf^2 - b4·xf^2
      coeff4 := da2b2·xf·yf
      vfx := coeff3·(xf - xj) + coeff4·(yj - yf)
      vfy := coeff3·(yj - yf) + coeff4·(xj - xf)
      xj := xf
      yj := yf
      vjx := vfx
      vjy := vfy
      n := n + 1
```

Figure 2. *Implementation of the procedure that traces orbits in an elliptical billiard, with rational formulae, in* Derive*, where duplicate repetitive operations are avoided as much as possible. If one has installed* Derive. See [S2] ABeJMT_2.

8. A practical case

For brevity sake only one example is worked out, similar to the Case 4.1 investigated in [1].

The equation of the contour of the chosen elliptical billiard is:

$$20\cdot x^2 + 60\cdot y^2 = 45 ,$$

where the semi-axes of the ellipse are: $a = \frac{3}{2}$ and $b = \frac{\sqrt{3}}{2}$.

The chosen starting points are:

$$P_0 = (-\tfrac{3\sqrt{3}}{4}, -\tfrac{\sqrt{3}}{4}), \quad P_1 = (\tfrac{3}{2}, 0),$$

so that the initial values of the variables for the algorithm are all algebraic numbers:

$$x_0 = -\tfrac{3\sqrt{3}}{4}, \quad y_0 = -\tfrac{\sqrt{3}}{4}, \quad v_{0x} = \tfrac{3}{4}(2+\sqrt{3}), \quad v_{0y} = \tfrac{\sqrt{3}}{4}.$$

The obtained orbit is plotted with Derive in Figure 3.

As an example, for the vertex $P(12)$ the algorithm implemented in Derive gives:

$$P(12) = \left[\frac{h_{12x}}{l_{12x}} - \frac{m_{12x}}{n_{12x}}\sqrt{3}, -\frac{h_{12y}}{l_{12y}} - \frac{m_{12y}}{n_{12y}}\sqrt{3}\right],$$

where:

$$h_{12x} = 46079347323825811716885760381836017418,$$
$$l_{12x} = 11981952408454099019611025980552171764 1,$$
$$m_{12x} = 138800698089762041778377411654839156155,$$
$$n_{12x} = 479278096338163960784441039222086870564,$$
$$h_{12y} = 93586129690918790287635648301246982922,$$
$$l_{12y} = 119819524084540990196110259805521717641,$$
$$m_{12y} = 227806036257395569770374626537175930 65,$$
$$n_{12y} = 479278096338163960784441039222086870564.$$

Even though it happens that: $l_{12y} = l_{12x}$, $n_{12y} = n_{12x}$, this is not to be expected for other vertices.

In order to verify that $P(12) \in$ E, that is to verify that $20 \cdot x_{P(12)}^2 + 60 \cdot y_{P(12)}^2 = 45$, one expands:

$$20\left(\frac{h_{12x}}{l_{12x}} - \frac{m_{12x}}{n_{12x}}\sqrt{3}\right)^2 = 20\left(\frac{h_{12x}}{l_{12x}}\right)^2 + 60\left(\frac{m_{12x}}{n_{12x}}\right)^2 - 40\sqrt{3}\left(\frac{h_{12x}}{l_{12x}} \cdot \frac{m_{12x}}{n_{12x}}\right),$$

$$60\left(\frac{h_{12y}}{l_{12y}} + \frac{m_{12y}}{n_{12y}}\sqrt{3}\right)^2 = 60\left(\frac{h_{12y}}{l_{12y}}\right)^2 + 180\left(\frac{m_{12y}}{n_{12y}}\right)^2 + 120\sqrt{3}\left(\frac{h_{12y}}{l_{12y}} \cdot \frac{m_{12y}}{n_{12y}}\right).$$

The sum of the double products gives:

$$\frac{40\sqrt{3}}{l_{12x} \cdot n_{12x}}\left(-h_{12x} \cdot m_{12x} + 3 \cdot h_{12y} \cdot m_{12y}\right),$$

and it is easy to verify with Derive itself that:

$$-h_{12x} \cdot m_{12x} + 3 \cdot h_{12y} \cdot m_{12y} = 0.$$

The sums of the similar terms are:

$$20\left(\frac{h_{12x}}{l_{12x}}\right)^2 + 60\left(\frac{h_{12y}}{l_{12y}}\right)^2 = \frac{20}{l_{12_x}^{\;2}}\left(h_{12_x}^{\;2} + 3h_{12_y}^{\;2}\right),$$

$$60\left(\frac{m_{12x}}{n_{12x}}\right)^2 + 180\left(\frac{m_{12y}}{n_{12y}}\right)^2 = \frac{60}{n_{12_x}^{\;2}}\left(m_{12_x}^{\;2} + 3m_{12_y}^{\;2}\right).$$

At last it is easy to verify with Derive itself that:

$$\frac{20}{l_{12_x}^{\;2}}\left(h_{12_x}^{\;2} + 3h_{12_y}^{\;2}\right) + \frac{60}{n_{12_x}^{\;2}}\left(m_{12_x}^{\;2} + 3m_{12_y}^{\;2}\right) = 45,$$

which proves that $P(12) \in$ E exactly!

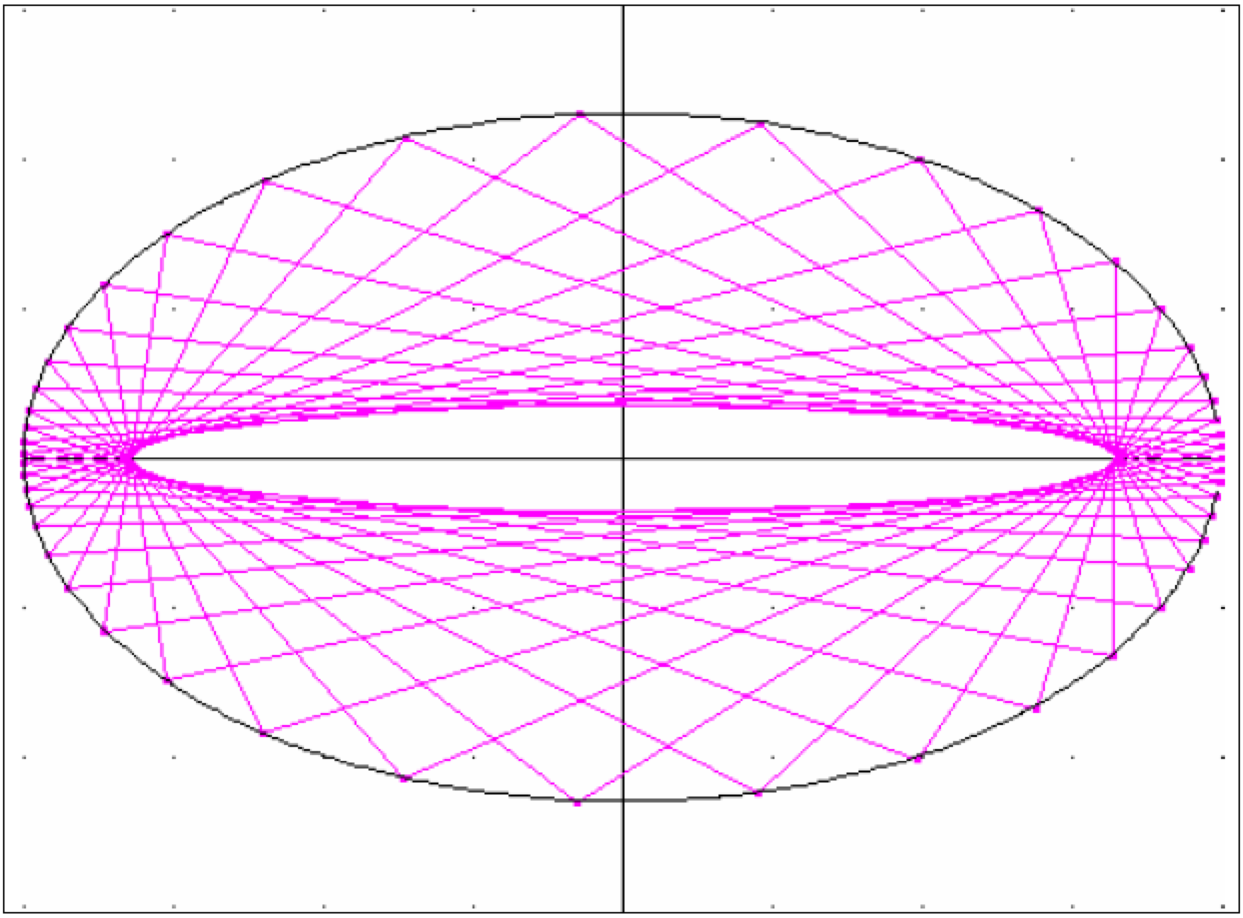

Figure 3. *Orbit in an ellipse with semi-axes* $a = \frac{3}{2}$, $b = \frac{\sqrt{3}}{2}$ *and equation* $20\cdot x^2 + 60\cdot y^2 = 45$, *built with starting points* $P_0 = (-\frac{3\sqrt{3}}{4}, -\frac{\sqrt{3}}{4})$ *and* $P_1 = (\frac{3}{2}, 0)$ *and consequently* $\mathbf{v}_0\ (\frac{3}{4}(2+\sqrt{3}), \frac{\sqrt{3}}{4})$. *The orbit is plotted with* Derive. *In the orbit, beyond* P_0 *there are* 47 *other vertices. Subsequent vertices are nearly graphically indistinguishable from the first* 48 *vertices already plotted and the orbit gives the impression of being closed on itself, but this is misleading.* See [S3] ABeJMT_3 .

9. The number of ciphers of the vertices of the trajectory

As anticipated, there is a price to pay in order to achieve exact results. In the case of the above worked out example, for instance, the output of the algorithm implemented in Derive for the vertex $P(141)$, which exactly satisfies the equation of the ellipse, appears structured in the following form:

$$P(141) = \left[s_{x_r}\frac{h_x}{l_x} + s_{x_i}\frac{m_x}{n_x}\sqrt{3}, s_{y_r}\frac{h_y}{l_y} + s_{y_i}\frac{m_y}{n_y}\sqrt{3}\right];\ s_{x_r}, s_{x_i}, s_{y_r}, s_{y_i} = +/-;\ h_x, l_x, m_x, n_x, h_y, l_y, m_y, n_y > 0;$$

where the parameters h_x, l_x, m_x, n_x, h_y, l_y, m_y, n_y are integer numbers with too many ciphers to be reproduced here. The parameters and the signs s_{x_r}, s_{x_i}, s_{y_r}, s_{y_i} are described in the following tables:

Sign	value	Sign	value
s_{x_r}	+	s_{y_r}	-
s_{x_i}	-	s_{y_i}	-

Parameter	number of ciphers	Parameter	number of ciphers
h_x	6258	h_y	6258
l_x	6261	l_y	6260
m_x	6260	m_y	6260
n_x	6260	n_y	6260

The points P_n generated by the algorithm all exactly satisfy the equation of the ellipse. It is this property that ensures that the orbits are accurate. The algorithm proceeds using exclusively the exact points P_n . Taking into account graphical representations, however, it is not necessary to use the exact points P_n , one can resort to replacing them with their best rational approximations, which will be called Q_n and are much more parsimonious in terms of their digit numbers.

Technically, with Derive, once an exact point P_n has been determined, its best rational approximation is obtained with the instruction "`Qn := APPROX(Pn)`" and the commands `SIMPLIFY`, `BASIC` or the keys "Ctrl + Enter".

As an example, the best rational approximation of P_{12} is $Q_{12} = \left[-\frac{551801}{4714824}, -\frac{5754503}{6665046}\right]$. The measure of the goodness of the approximation can be appreciated noting that:

$$20\cdot x_{Q_{12}}{}^2 + 60\cdot y_{Q_{12}}{}^2 - 45 = \frac{152227615207685}{6857641558647079953794064} \; \{ \; 1.077694655\cdot 10^{-11} ,$$

and that:

$$APPROX(P_{12} - Q_{12}) = [-\,2.5958118439491633652\cdot 10^{-13}, 2.2598496772509567904\cdot 10^{-13}] .$$

As another example, the best rational approximation of P_{141} is Q_{141} = [-7074989/5424792, -508207/1187916] . The goodness of the approximation can be appreciated noting that:

$$20\cdot x_{Q_{141}}{}^2 + 60\cdot y_{Q_{141}}{}^2 - 45 = -\,\frac{1160916055975}{26209666271895883600398 4} \; \{ \; -\,4.441868815\cdot 10^{-11} .$$

It would not be practicable to present tables with the coordinates of the exact points P_n but it is easy to give tables for their rational approximations Q_n. For reasons of space, we urge the reader to consult the Supplementary Electronic Materials, where the coordinates of the points Q_n for $n = 0$ to $n = 100$ are presented. Given are also the values of the polynomial $Q(x, y) = 20\cdot x^2 + 60\cdot y^2 - 45$, whose order of magnitude are typically 10^{-12}, 10^{-11} or 10^{-10}, proof of the proximity of the points Q_n to the contour of the considered elliptical billiard, whose equation is $20\cdot x^2 + 60\cdot y^2 = 45$.

10. The question of the periodicity of the trajectory

As stated in [1], one finds out that the point P_{47} is the first vertex of the orbit which falls in the near proximity of the initial vertex, namely the point P_0 , from which it is almost indistinguishable. In fact, the Euclidean distance of the two points is very small:

$$d(P_{47}, P_0) = \sqrt{(x_{P_{47}} - x_{P_0})^2 + (y_{P_{47}} - y_{P_0})^2} = \sqrt{(167967/96941857)^2 + (41303/23774278)^2} ,$$

which, truncated to 20 decimal places, gives:

$$d(P_{47}, P_0) \; \{ \; 0.00245363083189749328 .$$

This property will be indicated by the notation $P_{47} \approx P_0$.

Also, the Euclidean distance of their rational substitutes, Q_0 = [- 1091609/840321, - 489061/1129438] and Q_{47} = [- 1411374/1085029, - 945702/2192803], is negligible:

$$d(Q_{47}, Q_0) = \sqrt{((x_{Q_{47}} - x_{Q_0})^2 + (y_{Q_{47}} - y_{Q_0})^2)} ,$$

$$d(Q_{47}, Q_0) = \sqrt{((1579789393/911772654309)^2 + (4302652507/2476635034714)^2)} ,$$

which, truncated to 20 decimal places, gives: $d(Q_{47}, Q_0)$ { 0.00245363083189778905 .

As before, the fact will be indicated by the notation $Q_{47} \approx Q_0$. It is worth noting that $d(P_{47}, P_0)$ coincides with $d(Q_{47}, Q_0)$ up to the 15th decimal place. The subsequent points which fall in the near proximity of the starting point P_0 are P_{94} and P_{141}: $P_{94} \approx P_{47} \approx P_0$, then $P_{141} \approx P_{94} \approx P_{47} \approx P_0$.

In turn, the vertices which follow the point P_{47} fall in the near proximity of the vertices which follow the starting point P_0. It happens in fact that $P_{48} \approx P_1$, $P_{49} \approx P_2$, $P_{50} \approx P_3$ and so on:

$d(P_{48}, P_1) = \sqrt{((- 900842143\cdot 10^{-15}/5)^2 + (- 4244625531\cdot 10^{-13})^2)}$ { 0.00042446259133736742502 ,

$d(P_{49}, P_2) = \sqrt{((109145033\cdot 10^{-8}/625)^2 + (1741644141\cdot 10^{-12})^2)}$ { 0.0024663656866721582463 ,

$d(P_{50}, P_3) = \sqrt{((- 1431872129\cdot 10^{-11}/5)^2 + (980930373\cdot 10^{-11}/5)^2)}$ { 0.0034713007305509626617 .

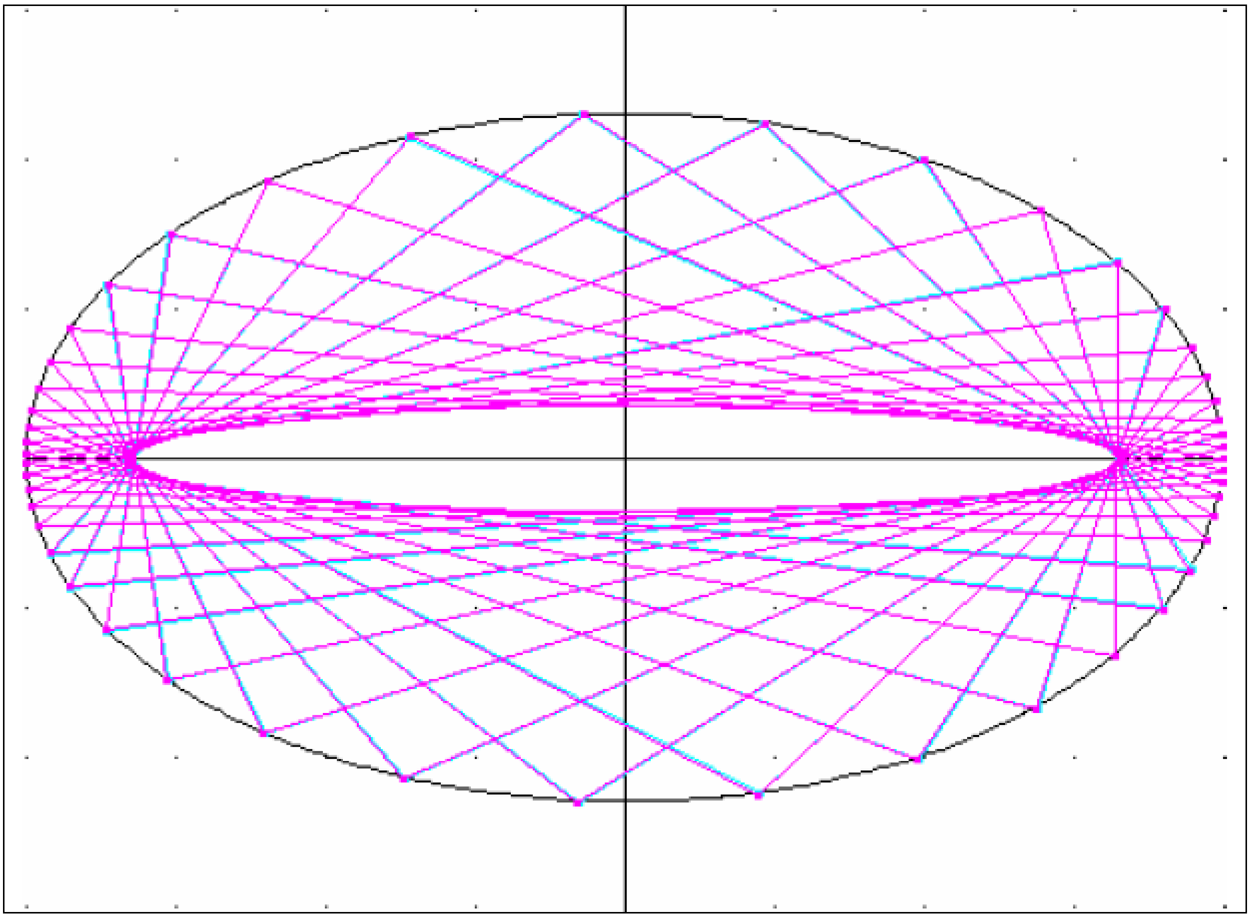

Figure 4. *Orbit in an ellipse with semi-axes* $a = \frac{3}{2}$, $b = \frac{\sqrt{3}}{2}$ *and equation* $20\cdot x^2 + 60\cdot y^2 = 45$, *built with starting points* $P_0 = (-\frac{3\sqrt{3}}{4}, -\frac{\sqrt{3}}{4})$ *and* $P_1 = (\frac{3}{2}, 0)$, *like in* Figure 3. *The orbit is plotted with* Derive. *In the orbit, beyond* P_0 *there are* 94 *other vertices. In pale blue are the rays connecting the vertices from* P_0 *to* P_{47}, *in purple are the rays connecting the vertices from* P_{47} *to* P_{94}. See [S4]

ABeJMT_4. In the Supplementary Electronic Materials, [S6], the reader will find tables with the differences of the coordinates of other pairs of neighbouring true points P_n . Taking into account the quadrants in which the points P_n fall and the signs of the differences of the corresponding coordinates of selected pairs of points P_n , a meticulous analysis shows that, counterclockwise, for any pair of nearly coinciding vertices $P_{47+n} \approx P_n$, the vertices P_{47+n} not only do not overlap the points P_n but lag a little behind them. The orbit is therefore only quasi-periodic: it seems to repeat itself but, with each revolution, it stays behind a little bit, even though, graphically, that characteristic is not easy to observe. The sequence of incident and reflected rays which connect the consecutive points from P_0 to P_{47} constitutes a first "almost closed in on itself" cycle. Similarly, the sequence of rays connecting the consecutive points from P_{47} to P_{94} constitutes a second such cycle. Using different colors for the two consecutive cycles, as in Figure 4, helps to visualize the precession-like phenomenon.

It would be interesting to find out if the orbit can finally close on itself, and after how many revolutions, but obviously a very powerful computer would be needed.

11. A completely rational example

If, with semi-axes $a = 40$ and $b = 30$, the equation of the contour of the elliptical billiard is:

$$\frac{x^2}{1600} + \frac{y^2}{900} = 1 \ ,$$

and the chosen starting point on the ellipse is:

$$P_0 = (-32, -18) \ ,$$

and the starting direction is:

$$v_0 = (236816, 11958687) \ ,$$

than the initial values of the variables for the algorithm are all rational numbers:

$$x_0 = -32 \ , \quad y_0 = -18 \ , \quad v_{0x} = 236816 \ , \quad v_{0y} = 11958687 \ .$$

The graphical representation of the orbit obtained with these initial rational values is like the one shown in Figure 3. The hyperlink to the Derive Worksheet is [S5] ABeJMT_5.

12. Concluding remarks

Failure to exploit all available resources can lead to false conclusions. Is it true that if there exists a closed polygon inscribed in an ellipse and circumscribing a confocal ellipse then there are infinitely many closed polygons which behave like the original one (the famous Poncelet's Closure Theorem [6]) but non-trivial numerical examples are nowhere to be found in the literature. I would be pleased to find one. I still consider this an open line of research and hope that some teachers will find the material presented here appropriate to inspire their brightest high school and college students to take up the challenge.

13. Supplementary Electronic Materials

[S1] Derive file, ABeJMT_1.
[S2] Derive file, ABeJMT_2.

[S3] Derive file, ABeJMT_3.
[S4] Derive file, ABeJMT_4.
[S5] Derive file, ABeJMT_5.
[S6] TABLES for Light Rays Reflected inside Ellipses.pdf.
[S7] Mathematica Note file, gl_gorni.nb.

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14. References

[1] Dávila-Rascón, G. and Yang, W.C., Investigating the Reflections of Light Rays Inside Ellipses with GeoGebra, Maxima and Maple, *The Electronic Journal of Mathematics and Technology*, **13** (3): 190-218, 2020.
[2] Lynch P., Integrable elliptic billiards and ballyards, *Eur. J. Phys*. **41** (1), 2020.
[3] https://en.wikipedia.org/wiki/Derive_(computer_algebra_system)
[4] For example, using the Search Service in http://vetusware.com/
[5] http://www.austromath.at/dug/
[6] (1813–2013): Current Advances, Bulletin (New Series) of the American Mathematical Society, Vol. 53, No. 3 (2014), pp. 373-445.